

ON THE ASYMPTOTICS OF THE α -FAREY TRANSFER OPERATOR

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ABSTRACT. We study the asymptotics of iterates of the transfer operator for non-uniformly hyperbolic α -Farey maps. We provide a family of observables which are Riemann integrable, locally constant and of bounded variation, and for which the iterates of the transfer operator, when applied to one of these observables, is not asymptotic to a constant times the wandering rate on the first element of the partition α . Subsequently, sufficient conditions on observables are given under which this expected asymptotic holds. In particular, we obtain an extension theorem which establishes that, if the asymptotic behaviour of iterates of the transfer operator is known on the first element of the partition α , then the same asymptotic holds on any compact set bounded away from the indifferent fixed point.

1. INTRODUCTION

Expanding maps of the unit interval have been widely studied in the last decades and the associated transfer operators have proven to be of vital importance in solving problems concerning the statistical behaviour of the underlying interval maps [3, 18].

In recent years an increasing amount of interest has developed in maps which are expanding everywhere except on an unstable fixed point (that is, an indifferent fixed point) at which trajectories are considerably slowed down. This leads to an interplay of chaotic and regular dynamics, a characteristic of intermittent systems [20, 22]. From an ergodic theory viewpoint, this phenomenon leads to an absolutely continuous invariant measure having infinite mass. Therefore, standard methods of ergodic theory cannot be applied in this setting; indeed it is wellknown that Birkhoff's ergodic theorem does not hold under these circumstances, see for instance [1].

In this paper we will be concerned with α -Farey maps, see Figure 1. These maps are of great interest since they are piecewise linear and expanding everywhere except for at the indifferent fixed point where they have (right) derivative one. This makes the α -Farey maps a simple model for studying the physical phenomenon of intermittency [20]. Moreover, an induced version of the α -Farey maps are given by the α -Lüroth maps introduced in [17], which have significant meaning in number theory, see for instance [4, 15].

Thaler [24] was the first to discern the asymptotics of the transfer operator of a class of interval maps preserving an infinite measure. This class of maps, to which the α -Farey maps do not belong, have become to be known as Thaler maps. In an effort to generalise this work, by combining renewal theoretical arguments and functional analytic techniques, a new approach to estimate the decay of correlation of a dynamical system was achieved by Sarig [21]. Subsequently, Gouëzel [10, 11, 12] generalised these methods. Using these ideas and employing the methods of Garsia and Lamperti [9], Erickson [8] and Doney [7], recently Melbourne and Terhesiu [19, Theorem 2.1 to 2.3] proved a landmark result on the asymptotic rate of convergence of iterates of the induced transfer operator and showed that these result can be applied to Gibbs-Markov maps, Thaler maps, AFN maps, and Pomeau-Manneville maps. Thus, the question which naturally arises is, whether this asymptotic rate can be related to the asymptotic rate of convergence of iterates of the actual

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transfer operator. The results of this paper give some positive answers to this question for δ -expansive α -Farey maps.

As mentioned above, in this paper, we will consider the α -Farey map $F_\alpha: [0, 1] \rightarrow [0, 1]$, which is given for a countable infinite partition $\alpha := \{A_n : n \in \mathbb{N}\}$ of $(0, 1)$ by non-empty intervals A_n . It is assumed throughout that the atoms of α are ordered from right to left, starting with A_1 , and that these atoms only accumulate at zero. Further, we assume that A_n is right-open and left-closed, for all natural numbers n . We define the α -Farey map $F_\alpha: [0, 1] \rightarrow [0, 1]$ by

$$F_\alpha(x) := \begin{cases} (1-x)/a_1 & \text{if } x \in \bar{A}_1 := A_1 \cup \{1\}, \\ a_{n-1}(x-t_{n+1})/a_n + t_n & \text{if } x \in A_n, \text{ for } n \geq 2, \\ 0 & \text{if } x = 0, \end{cases}$$

where a_n is equal to the Lebesgue measure $\lambda(A_n)$ of the atom $A_n \in \alpha$ and $t_n := \sum_{k=n}^{\infty} a_k$ denotes the Lebesgue measure of the n -th tail $\bigcup_{k=n}^{\infty} A_k$ of α , see Figure 1. Throughout, we will assume that the partition α satisfies the condition that the sequence $(t_n)_{n \in \mathbb{N}}$ is not summable. For $\delta \in (0, 1]$, an α -Farey map F_α is said to be δ -expansive if the sequence $(a_n)_{n \in \mathbb{N}}$ is regularly varying of order $-(1+\delta)$, that is, if there exists a slowly varying function $l: \mathbb{R} \rightarrow \mathbb{R}$ such that $a_n = \delta l(n)n^{-(1+\delta)}$, for all $n \in \mathbb{N}$. (Recall that $l: [a, \infty) \rightarrow \mathbb{R}$ is called a *slowly varying function*, if it is measurable, locally Riemann integrable and $\lim_{x \rightarrow \infty} l(\eta x)/l(x) = 1$, for each $\eta > 0$ and for some $a \in \mathbb{R}$, see [6, 23] for further details.) In this situation, [6, Theorem 1.5.10] implies that

$$\lim_{n \rightarrow \infty} \frac{l(n)n^{-\delta}}{t_n} = \lim_{n \rightarrow \infty} \frac{l(n)n^{-\delta}}{\sum_{k=n}^{\infty} a_k} = \lim_{n \rightarrow \infty} \frac{l(n)n^{-\delta}}{\sum_{k=n}^{\infty} \delta l(k)k^{-(1+\delta)}} = 1.$$

Therefore, the Lebesgue measure of the n -th tail of α is asymptotic to a regularly varying function of order $-\delta$. Thus, δ -expansive implies expansive of order δ in the sense of [15]. However, an expansive α -Farey map of order δ is not necessarily δ -expansive.

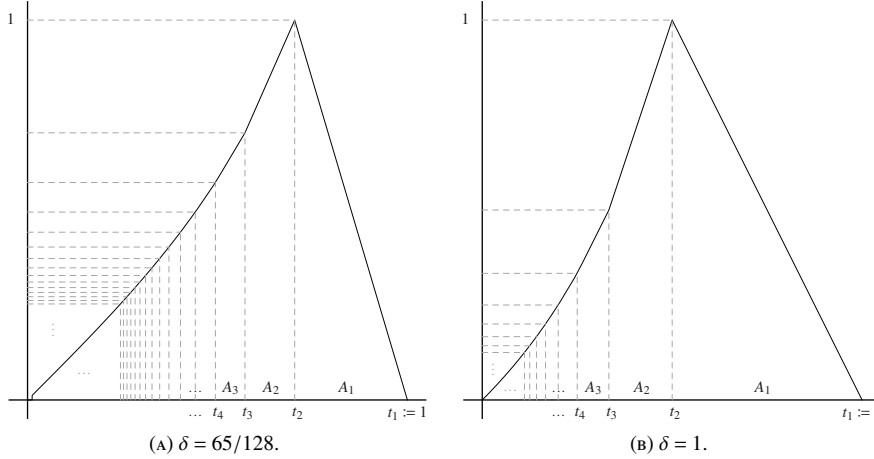


FIGURE 1. The α -Farey map, where $t_n = n^{-\delta}$, for all $n \in \mathbb{N}$.

Throughout, let μ_α denote the F_α -invariant measure which is determined by

$$h_\alpha := \frac{d\mu_\alpha}{d\lambda} = \sum_{n \in \mathbb{N}} \frac{t_n}{a_n} \mathbb{1}_{A_n} \quad (1)$$

and let \mathcal{B} denote the Borel σ -algebra of $[0, 1]$. Here and in the sequel, for a given Borel set $B \in \mathcal{B}$, we let $\mathbb{1}_B$ denote the indicator function on B . It is verified in [15] that, since

the sequence $(a_n)_{n \in \mathbb{N}}$ is regularly varying of order $-(1 + \delta)$, the map F_α is conservative, ergodic and measure preserving on the infinite and σ -finite measure space $([0, 1], \mathcal{B}, \mu_\alpha)$. The dynamical system $([0, 1], \mathcal{B}, \mu_\alpha, F_\alpha)$ will be referred to as a α -Farey system.

Following the definitions and notations of [5], throughout, we let $\mathcal{L}_{\mu_\alpha}^1([0, 1])$ (respectively $\mathcal{L}_\lambda^1([0, 1])$) denote the class of measurable functions f with domain $[0, 1]$ for which $|f|$ is μ_α -integrable (respectively λ -integrable), and for $f \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ (respectively $\mathcal{L}_\lambda^1([0, 1])$), define $\|f\|_{\mu_\alpha}$ (respectively $\|f\|_\lambda$) by

$$\|f\|_{\mu_\alpha} := \int |f| d\mu_\alpha \quad \left(\text{respectively } \|f\|_\lambda := \int |f| d\lambda \right).$$

Further, for a given measurable function w , we set $\|w\|_\infty := \sup_{x \in [0, 1]} |w(x)|$.

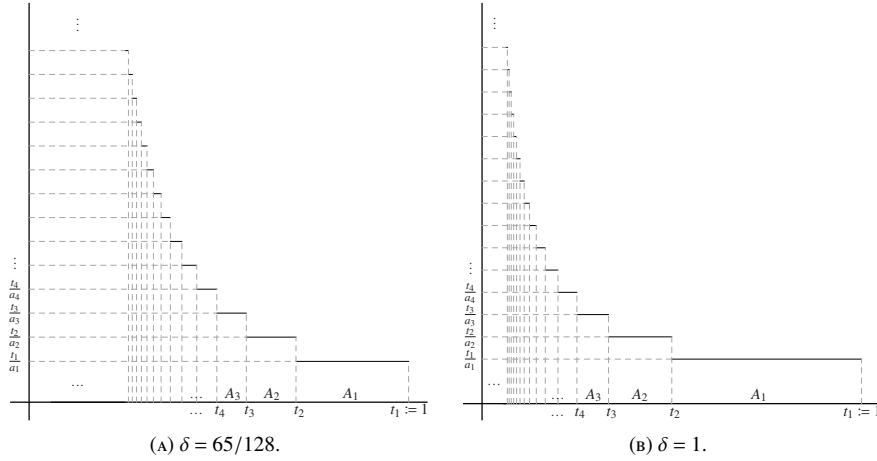


FIGURE 2. Plot of the density function h_α for the α -Farey map, where $t_n = n^{-\delta}$ for all $n \in \mathbb{N}$.

The α -Farey transfer operator $\widehat{F}_\alpha: \mathcal{L}_{\mu_\alpha}^1([0, 1]) \rightarrow \mathcal{L}_{\mu_\alpha}^1([0, 1])$ is the positive linear operator given by

$$\widehat{F}_\alpha(v) := \sum_{n \in \mathbb{N}} \left(\frac{t_{n+1}}{t_n} v \circ F_{\alpha,0} + \left(1 - \frac{t_{n+1}}{t_n} \right) v \circ F_{\alpha,1} \right) \cdot \mathbb{1}_{A_n}, \quad (2)$$

where $F_{\alpha,0} := (F_\alpha|_{[0,t_2]})^{-1}$ and $F_{\alpha,1} := (F_\alpha|_{[t_2,1]})^{-1}$ refer to the inverse branches of F_α . In particular, for all $v \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ and all measurable functions w with $\|w\|_\infty < \infty$,

$$\int \widehat{F}_\alpha(v) \cdot w d\mu_\alpha = \int v \cdot w \circ F_\alpha d\mu_\alpha. \quad (3)$$

(The above equality is a direct consequence of [15, Lemma 2.5].) Note that the equality given in (3) is the usual defining relation for the transfer operator of F_α . However, the relation in (3) only determines values of the transfer operator of F_α applied to an observable μ_α -almost everywhere. Thus the α -Farey transfer operator is a version of the transfer operator of F_α .

In order to state our main theorems, we will also require the following function spaces. We let $\phi: \overline{A}_1 \rightarrow \mathbb{N} \cup \{+\infty\}$ denote the first return time, given by $\phi(y) := \inf\{n \in \mathbb{N} : F_\alpha^n(y) \in \overline{A}_1\}$, and we write $\{\phi = n\} := \{y \in A_1 : \phi(y) = n\}$. Let \mathcal{B}_α denote the countable-infinite partition $\{\{\phi = n\} : n \in \mathbb{N}\}$ of A_1 and let \mathcal{B}_α denote the set of functions with domain $[0, 1]$ that are supported on a subset of \overline{A}_1 and which have finite $\|\cdot\|_{\mathcal{B}_\alpha}$ -norm, where $\|\cdot\|_{\mathcal{B}_\alpha} := \|\cdot\|_\infty + D_\alpha(\cdot)$

and where

$$D_\alpha(f) := \sup_{a \in \beta_\alpha} \sup_{x \neq y \in a} \frac{|f(x) - f(y)|}{|x - y|}.$$

In particular, if $f \in \mathcal{B}_\alpha$, then f is Lipschitz continuous on each atom of β_α , zero outside of \bar{A}_1 and bounded (everywhere). We then define

$$\mathcal{A}_\alpha := \left\{ v \in \mathcal{L}_{\mu_\alpha}^1([0, 1]) : \|v\|_\infty < \infty \text{ and } \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}) \in \mathcal{B}_\alpha \text{ for all } n \in \mathbb{N} \right\}.$$

For examples of observables belonging to \mathcal{A}_α , we refer the reader to Example 4.4 and the discussion succeeding our main results, Theorems 1.1 and 1.3. Let us also recall from [15] that the *wandering rate* of F_α is given by

$$w_n = w_n(F_\alpha) := \mu_\alpha \left(\bigcup_{k=0}^{n-1} F_\alpha^{-k}(A_1) \right) = \mu_\alpha \left(\bigcup_{k=1}^n A_k \right) = \sum_{k=1}^n t_k.$$

Further, as we will see in (12), if $\delta \in (0, 1)$ and if the given α -Farey system is δ -expansive, then the wandering rate is regularly varying of order $1 - \delta$. Also, in the case that $\delta = 1$, if

$$(w_n / w_{\lceil n \cdot w_n^{-2} \rceil})_{n \in \mathbb{N}}$$

is a bounded sequence, then we say that the wandering rate w_n is *moderately increasing*. Here and in the sequel for $r \in \mathbb{R}$ we let $\lceil r \rceil$ denote the smallest integer greater than or equal to r .

With the above preparations, we are now in a position to state the main results, Theorems 1.1 and 1.3. Theorem 1.1 provides mild conditions under which the asymptotic behavior of the iterates of an α -Farey transfer operator *restricted to* A_1 can be extended to all of $(0, 1]$ and is used in our proof of Theorem 1.3. (Note that, by (12), any δ -expansive α -Farey system satisfies the requirements of Theorem 1.1.) One of the facets of Theorem 1.3 is that it gives sufficient conditions on observables which guarantee that iterates of an α -Farey transfer operator applied to such an observable is asymptotic to a constant times the wandering rate. These results complement [19, Theorem 10.5] and show that additional assumptions are required in [19, Theorem 10.4]. Namely, in the case that $\delta = 1$, we show that the statement of [19, Theorem 10.4] holds true, with the additional assumption that the wandering rate is moderately increasing (Theorem 1.3(i)); for $\delta \in (1/2, 1)$, we provide an example which demonstrates that additional requirements are necessary for the expected convergence (Theorem 1.3(iii)) and provide sufficient conditions (Theorem 1.3(ii)).

Theorem 1.1. *For an α -Farey system $([0, 1], \mathcal{B}, \mu_\alpha, F_\alpha)$ for which the wandering rate satisfies the condition $\lim_{n \rightarrow \infty} w_n / w_{n+1} = 1$, we have that, if $v \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ satisfies*

$$\lim_{n \rightarrow +\infty} w_n \widehat{F}_\alpha^n(v) = \Gamma_\delta \int v d\mu_\alpha$$

uniformly on \bar{A}_1 , then the same holds on any compact subset of $(0, 1]$. The same statement holds when replacing uniform convergence by almost everywhere uniform convergence.

Remark 1.2. For $\delta \in (0, 1)$, a δ -expansive α -Farey system has wandering rate satisfying $\lim_{n \rightarrow \infty} w_n / w_{n+1} = 1$.

Theorem 1.3. *Let $([0, 1], \mathcal{B}, \mu_\alpha, F_\alpha)$ be a δ -expansive α -Farey system.*

(i) Let $\delta = 1$ and assume that the wandering rate is moderately increasing. If $v \in \mathcal{A}_\alpha$ and if

$$\sum_{k=1}^{\infty} \left\| \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_\infty < +\infty, \quad (4)$$

then uniformly on compact subsets of $(0, 1]$,

$$\lim_{n \rightarrow \infty} w_n \widehat{F}_\alpha^n(v) = \int v d\mu_\alpha. \quad (5)$$

- (ii) For $\delta \in (1/2, 1]$, if $v \in \mathcal{L}_\lambda^1([0, 1])$ with $|v(1)|$ bounded and if
- (a) the sequence $(D_\alpha(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n})))_{n \in \mathbb{N}}$ is bounded and
 - (b) there exist constants $c > |v(1)|$ and $\eta \in (0, \delta)$ with $\|v \cdot \mathbb{1}_{A_n}\|_\infty \leq cn^\eta$, for all $n \in \mathbb{N}$.
- then uniformly on compact subsets of $(0, 1]$,

$$\lim_{n \rightarrow \infty} w_n \widehat{F}_\alpha^n(v/h_\alpha) = \Gamma_\delta \int v/h_\alpha d\mu_\alpha. \quad (6)$$

Here, $\Gamma_\delta := (\Gamma(1 + \delta)\Gamma(2 - \delta))^{-1}$ and Γ denotes the Gamma function.

- (iii) For $\delta \in (1/2, 1)$, there exists a positive, locally constant, Riemann integrable function $v \in \mathcal{A}_\alpha$ of bounded variation satisfying the inequality in (4), such that, for all $x \in \bar{A}_1$,

$$\liminf_{n \rightarrow \infty} w_n \widehat{F}_\alpha^n(v)(x) = \Gamma_\delta \int v d\mu_\alpha \quad \text{and} \quad \limsup_{n \rightarrow \infty} w_n \widehat{F}_\alpha^n(v)(x) = +\infty. \quad (7)$$

Remark 1.4. It is immediate that if $a_n = n^{-1}(n+1)^{-1}$, then $t_n = n^{-1}$ and $w_n \sim \ln(n)$, and that these parameters give rise to an example of an α -Farey system which satisfies the conditions of Theorem 1.3(i). Indeed there exist many examples of α -Farey systems for which the conditions of Theorem 1.3(i) are satisfied, but where the wandering rate behaves very differently to the function $n \mapsto \ln(n)$. Letting $\delta = 1$, as we will see in Lemma 2.6(iv), the sequence $(w_n)_{n \in \mathbb{N}}$ is slowly varying and $\lim_{n \rightarrow \infty} nt_n/w_n = 0$. We also have that

$$w_n = \sum_{j=1}^n t_j = n \sum_{j=n+1}^\infty a_j + \sum_{j=1}^n ja_j = nt_{n+1} + \sum_{j=1}^n ja_j.$$

Using this we deduce the following.

- (1) If $a_n = n^{-2}(\ln(n))^{-1/2}e^{(\ln(n))^{1/2}}$, then $t_n \sim n^{-1}(\ln(n))^{-1/2}e^{(\ln(n))^{1/2}}$ and $w_n \sim e^{(\ln(n))^{1/2}}$.
- (2) If $a_{n-16} = n^{-2}\kappa(n)e^{\ln(n)/\ln(\ln(n))}$, then $t_n \sim n^{-1}\kappa(n)e^{\ln(n)/\ln(\ln(n))}$ and $w_n \sim e^{\ln(n)/\ln(\ln(n))}$, where $\kappa(n) = (\ln(\ln(n)) - 1)(\ln(\ln(n)))^{-2}$.

Indeed the above two sets of parameters give rise to examples of 1-expansive α -Farey systems whose wandering rate is moderately increasing. Moreover,

$$\lim_{n \rightarrow \infty} \frac{\ln(n)}{e^{(\ln(n))^{1/2}}} = 0, \quad \lim_{n \rightarrow \infty} \frac{\ln(n)}{e^{\ln(n)/\ln(\ln(n))}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{e^{(\ln(n))^{1/2}}}{e^{\ln(n)/\ln(\ln(n))}} = 0,$$

demonstrating that two moderately increasing wandering rates, although they are all slowly varying, do not have to be asymptotic to each other nor to the function $n \mapsto \ln(n)$.

Remark 1.5. In the case that F_α is a 1-expansive α -Farey map, we have that the wandering rate w_n is a slowly varying function. We remark here that it is not the case that every slowly varying function is moderately increasing, namely, it is not the case that if $l: [0, \infty] \rightarrow \mathbb{R}$ is a slowly varying function, then the sequence

$$(l(n)/l(\lceil n \cdot l(n)^{-2} \rceil))_{n \in \mathbb{N}} \quad (8)$$

is bounded. For instance consider the following. Let $(c_k)_{k \in \mathbb{N}}$ be a decreasing sequence of positive real numbers which converge to zero and, for $k \in \mathbb{N}$, set

$$x_{k+1} = \frac{k}{2c_k^2} \quad \text{and} \quad b_{k+1} = (c_k - c_{k+1})x_{k+1} + b_k,$$

where $x_1 = b_1 = 0$. We define $m: [0, \infty) \rightarrow \mathbb{R}$ by

$$m(x) := c_k x + b_k,$$

for $x \in [x_k, x_{k+1}]$. The function $l: [1, \infty) \rightarrow \mathbb{R}$ defined by $l(x) := e^{m(\ln(x))}$ is, by construction, slowly varying. However, the sequence given in (8) is unbounded. (We are grateful to Fredrik Ekström for providing this example).

Remark 1.6. If in the definition of the norm $\|\cdot\|_{\mathcal{B}_\alpha}$, one replaces the norm $\|\cdot\|_\infty$ by the *essential supremum norm* $\|\cdot\|_{\text{ess sup}}$, then by appropriately adapting the proofs given in the sequel, one can obtain a proof of Theorem 1.3 where the uniform convergence on compact subsets of $(0, 1]$ is replaced by uniform convergence almost everywhere on compact subsets of $(0, 1)$.

Remark 1.7. The first part of the proof of Theorem 1.3 (i) and (ii) are inspired by the first paragraph in the proof of [19, Theorem 10.4].

The structure of this paper is as follows. In Section 2 we collect basic properties of α -Farey maps and their corresponding transfer operators. In Section 3 we provide a proof of Theorem 1.1. This proof is inspired by arguments originally presented in [14]. Then in Section 4 we present the proof of Theorem 1.3, breaking the proof into three constituent parts. In Section 4.1 we obtain part (i) and give explicit examples of observables satisfying the given properties. In Section 4.2 we prove part (ii), for explicit examples of observables which satisfy the pre-requests of Theorem 1.3 (ii) we refer the reader to Remark 1.9. Finally we conclude with Section 4.3 where part (iii) is proven using a constructive argument.

Before we conclude this section with a series of remarks, Remarks 1.8 to 1.10, in which we comment on how Theorem 1.3, and hence Theorem 1.1, complement the results obtained in [16, 24], we introduce the *Perron-Frobenius operator* $\mathcal{P}_\alpha: \mathcal{L}_\lambda^1([0, 1]) \rightarrow \mathcal{L}_\lambda^1([0, 1])$ which is defined by

$$\mathcal{P}_\alpha(f)(x) := \sum_{y \in F_\alpha^{-1}(x)} |F'_\alpha(y)|^{-1} f(y),$$

where F'_α denotes the right derivative of F_α and where $F'_\alpha(1) := -a_1^{-1}$. (Note, by construction, if F_α is δ -expansive, then the right derivative of F_α at zero is equal to one.) A useful relation between the operators \mathcal{P}_α and \widehat{F}_α is that

$$\widehat{F}_\alpha(f) = \mathcal{P}_\alpha(h_\alpha \cdot f) / h_\alpha. \quad (9)$$

We refer the reader to [15, p. 1001] for a proof of the equality in (9).

Remark 1.8. For certain interval maps $T: [0, 1] \rightarrow [0, 1]$ with two monotonically increasing, differentiable branches whose invariant measure has infinite mass and whose tail probabilities are regularly varying with exponent $-\delta \in [-1, 0)$, Thaler [24] discerned the precise asymptotic behaviour of iterates of the associated Perron-Frobenius operator \mathcal{P} , namely, that for all Riemann integrable functions u with domain $[0, 1]$, one has that

$$\lim_{n \rightarrow +\infty} w_n(T) \mathcal{P}^n(u) = \Gamma_\delta \left(\int u d\lambda \right) h \quad (10)$$

uniformly almost everywhere on compact subsets of $(0, 1]$. Here, h denotes the associated invariant density and $w_n(T)$ denotes the wandering rate of T . However, α -Farey maps do not fall into this class of interval maps. Using the relationship between the transfer and the Perron-Frobenius operator, Theorem 1.3 (ii) together with the assumption that the Banach space of functions of bounded variation with the norm $\|\cdot\|_{\text{ess sup}} + \text{Var}(\cdot)$ satisfies certain functional analytic conditions (namely, conditions (H1) and (H2) given in Section 2), show that Thaler's result can be extended to δ -expansive α -Farey maps. Results of this form have also been obtained in [27] for AFN maps. (Note, an α -Farey map is also not an AFN map.)

Remark 1.9. Kesseböhmer and Slassi [16] showed that for the classical Farey map the convergence given in (10) holds uniformly almost everywhere on $[1/2, 1]$ for convex C^2 -observables. Likewise, for a δ -expansive α -Farey map, Theorems 1.3 (ii) implies that if u is a convex C^2 -observable, then the convergence in (10) holds uniformly on compact subsets of $(0, 1]$. To see that a convex C^2 -observable satisfies the requirements of Theorem 1.3 (ii), one employs arguments similar to those used in Example 4.4 together with (1) and (14).

Remark 1.10. The consequences of Theorem 1.3 go even further, in that for a δ -expansive α -Farey map, we are able to obtain that the convergence given in (10) holds uniformly on compact subsets of $(0, 1]$, for certain non-Riemann integrable observables which are not necessarily bounded. For instance, if v is an observable such that $v \cdot \mathbb{1}_{A_n} = n^\eta \mathbb{1}_{A_n}$, $v(0) = 0$ and $v(1) = 1$, for some $\eta \in (0, \delta)$, then, as we will see in Lemma 2.3, since $0 \leq F^{n-1}(\mathbb{1}_{A_n}) \leq t_n \mathbb{1}_{A_1}$, this observable fulfils the conditions of Theorem 1.3 (ii) and it is neither Riemann integrable nor is it bounded.

Notation 1.11. We use the symbol \sim between the elements of two sequences of real numbers $(b_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ to mean that the sequences are asymptotically equivalent, namely that $\lim_{n \rightarrow +\infty} b_n/c_n = 1$. We use the Landau notation $b_n = o(c_n)$, if $\lim_{n \rightarrow +\infty} b_n/c_n = 0$. The same notation is used between two real-valued function f and g , defined on the set of real numbers \mathbb{R} , positive real numbers \mathbb{R}^+ , natural numbers \mathbb{N} or non-negative integers \mathbb{N}_0 . Specifically, if $\lim_{x \rightarrow +\infty} f(x)/g(x) = 1$, then we will write $f \sim g$, and if $\lim_{x \rightarrow +\infty} f(x)/g(x) = 0$, then we will write $f \in o(g)$.

2. THE α -FAREY SYSTEM

The map $G_\alpha: \bar{A}_1 \rightarrow \bar{A}_1$ defined by

$$G_\alpha(x) := \begin{cases} F_\alpha^{\phi(x)}(x) & \text{if } x \in A_1, \\ t_2 & \text{if } x = 1, \end{cases}$$

is called the *first return map* and it is well known that G_α is conservative, ergodic and measure preserving on $(\bar{A}_1, \mathcal{B}|_{\bar{A}_1}, \mu_\alpha|_{\bar{A}_1})$, see for instance [1, Propositions 1.4.8 and 1.5.3]. From this point on, we write μ_α for both μ_α and $\mu_\alpha|_{\bar{A}_1}$ and \mathcal{B} for both \mathcal{B} and $\mathcal{B}|_{\bar{A}_1}$. Also, throughout, unless otherwise stated, we assume that F_α is δ -expansive.

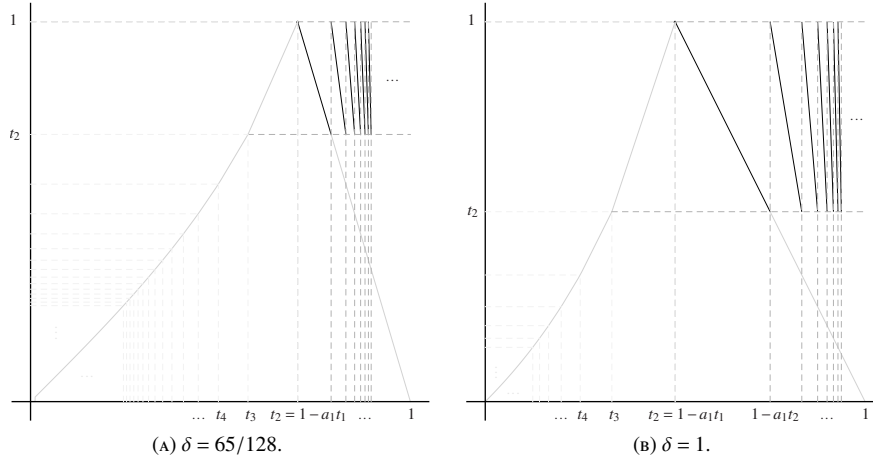


FIGURE 3. Plot of G_α , where $t_n = n^{-\delta}$ for all $n \in \mathbb{N}$.

We denote the open unit disk in \mathbb{C} by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, its closure by $\bar{\mathbb{D}}$ and its boundary by \mathbb{S} . Given $z \in \bar{\mathbb{D}}$, define $R_n, R(z): \mathcal{L}_{\mu_\alpha}^1([0, 1]) \rightarrow \mathcal{L}_{\mu_\alpha}^1([0, 1])$ by

$$R_n(v) := \widehat{F}_\alpha(v \cdot \mathbb{1}_{\{\phi=n\}}) = \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^n(v \cdot \mathbb{1}_{\{\phi=n\}}) \quad \text{and} \quad R(z) := \sum_{n \in \mathbb{N}} z^n R_n.$$

It is an easy exercise to show that $R(1)$ is a version of the transfer operator of the map G_α . Namely, for all $v \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ and all measurable functions w with $\|w\|_\infty$ finite, we have that

$$\int R(1)(v) \cdot w d\mu_\alpha = \int v \cdot w \circ G_\alpha \cdot \mathbb{1}_{\bar{A}_1} d\mu_\alpha. \quad (11)$$

We will see in Proposition 2.4 that $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$ is a Banach space, that the operators R_n and $R(1)$ map \mathcal{B}_α into itself and that the following properties are fulfilled.

- (H1):** There exists a constant $c > 0$ such that the operator $R_n : \mathcal{B}_\alpha \rightarrow \mathcal{B}_\alpha$ is a bounded linear operator with $\|R_n\|_{\text{op}} \leq c\mu(\{\phi = n\})$, for all $n \in \mathbb{N}$. (Here, the operator norm $\|\cdot\|_{\text{op}}$ is taken with respect to the Banach space $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$.)
- (H2):** (i) The operator $R(1)$ restricted to \mathcal{B}_α has a simple isolated eigenvalue at 1.
(ii) For each $z \in \overline{\mathbb{D}} \setminus \{1\}$, the value 1 is not in the spectrum of $R(z)|_{\mathcal{B}_\alpha}$.

A result that will be crucial in the proof of Theorem 1.3 is [19, Theorem 2.1]. In order to see how this result reads in our situation, note, for a δ -expansive α -Farey map, that $\mu(\{y \in A_1 : \phi(y) > n\}) = t_{n+1} \sim l(n)n^{-\delta}$, which is essential in the proof of [19, Theorem 2.1] given in [19]. Further, since $t_n \sim l(n)n^{-\delta}$ and $t_{n+1} < t_n$, for all $n \in \mathbb{N}$, Karamata's Tauberian Theorem for power series [6, Corollary 1.7.3] implies that, for $\delta \in (0, 1)$,

$$w_n \sim \bar{\Gamma}_\delta n^{1-\delta} l(n). \quad (12)$$

Here, $\bar{\Gamma}_\delta := \Gamma(1-\delta)/\Gamma(2-\delta)$.

Theorem 2.1 ([19, Theorem 2.1]). *Assuming the above setting, in particular that conditions (H1) and (H2) are satisfied, we have that*

$$\lim_{n \rightarrow +\infty} \sup_{v \in \mathcal{B}_\alpha : \|v\|_{\mathcal{B}_\alpha} = 1} \left\| w_n \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^n(v) - \Gamma_\delta \int v d\mu_\alpha \right\|_{\mathcal{B}_\alpha} = 0.$$

In the sequel, we will also use of the following auxiliary results, where we set $\Sigma := \{0, 1\}$ and for each $n \in \mathbb{N}$ and for each word $\omega := (\omega_1, \omega_2, \dots, \omega_n) \in \Sigma^n$ we let $F_{\alpha, \omega} : [0, 1] \rightarrow [0, 1]$ denote the function $F_{\alpha, \omega_1} \circ F_{\alpha, \omega_2} \circ \dots \circ F_{\alpha, \omega_n}$. If ω is equal to the empty word, then we set $F_{\alpha, \omega}$ to be equal to the identity map.

Lemma 2.2. *Let $F_\alpha : [0, 1] \rightarrow [0, 1]$ denote an arbitrary α -Farey map. For each $k \in \mathbb{N}$, we have that*

$$\widehat{F}_\alpha^k(u) = \sum_{n \in \mathbb{N}} \sum_{\omega \in \Sigma^k} c_{n, \omega} u \circ F_{\alpha, \omega} \cdot \mathbb{1}_{A_n},$$

where the constants $c_{n, \omega}$ are given recursively by

$$\begin{aligned} c_{n, (0)} &:= t_{n+1}/t_n, & c_{n, (\omega_1, \dots, \omega_k, 0)} &:= c_{n, (0)} c_{n+1, \omega}, \\ c_{n, (1)} &:= 1 - t_{n+1}/t_n, & c_{n, (\omega_1, \dots, \omega_k, 1)} &:= c_{n+1, (1)} c_{1, \omega}. \end{aligned} \quad (13)$$

In particular, letting $0_k := \underbrace{(0, 0, \dots, 0)}_{k\text{-times}}$, we have that $c_{1, 0_k} = t_{k+1}$, for each $k \in \mathbb{N}$.

Proof. We proceed by induction on k . The start of the induction is an immediate consequence of (2). Suppose that the statement is true for some $k \in \mathbb{N}$. We then have that

$$\begin{aligned} \widehat{F}_\alpha^{k+1}(u) &= \widehat{F}_\alpha(\widehat{F}_\alpha^k(u)) = \widehat{F}_\alpha \left(\sum_{n \in \mathbb{N}} \sum_{\omega \in \Sigma^k} c_{n, \omega} u \circ F_{\alpha, \omega} \cdot \mathbb{1}_{A_n} \right) \\ &= \sum_{m=1}^{\infty} \left(\sum_{n \in \mathbb{N}} \sum_{\omega \in \Sigma^k} \frac{t_{m+1}}{t_m} c_{n, \omega} u \circ F_{\alpha, \omega} \circ F_{\alpha, 0} \cdot \mathbb{1}_{A_n} \circ F_{\alpha, 0} \right. \\ &\quad \left. + \left(1 - \frac{t_{m+1}}{t_m} \right) c_{n, \omega} u \circ F_{\alpha, \omega} \circ F_{\alpha, 1} \cdot \mathbb{1}_{A_n} \circ F_{\alpha, 1} \right) \cdot \mathbb{1}_{A_m} \\ &= \sum_{m=1}^{\infty} \left(\sum_{\omega \in \Sigma^k} \frac{t_{m+1}}{t_m} c_{m+1, \omega} u \circ F_{\alpha, \omega} \circ F_{\alpha, 0} + \left(1 - \frac{t_{m+1}}{t_m} \right) c_{1, \omega} u \circ F_{\alpha, \omega} \circ F_{\alpha, 1} \right) \cdot \mathbb{1}_{A_m}. \end{aligned}$$

This completes the proof of (13). The remaining assertion is proven by a straight forward inductive argument, using the defining relations given in (13). \square

Lemma 2.3. *For each $n \in \mathbb{N}$, we have that*

$$\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n}) = t_n \mathbb{1}_{A_1} \quad (14)$$

and hence, by the definition of the norm, $\|\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n})\|_{\mathcal{B}_\alpha} = \|\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n})\|_\infty = t_n$.

Proof. For $n = 1$ the result is immediate. For $n \neq 1$, we have, by Lemma 2.2, that, on $[0, 1)$,

$$\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n}) = \sum_{k=1}^{\infty} c_{k,0_{n-1}} \mathbb{1}_{A_n} \circ F_{\alpha,0_{n-1}} \cdot \mathbb{1}_{A_k} = \sum_{k=1}^{\infty} c_{k,0_{n-1}} \mathbb{1}_{A_1} \cdot \mathbb{1}_{A_k} = t_n \mathbb{1}_{A_1}.$$

To complete the proof, we need to evaluate the function $\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n})$ at the point 1 for $n \geq 2$. By Lemma 2.2, we have that

$$\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n})(1) = \sum_{n \in \mathbb{N}} \sum_{\omega \in \Sigma} c_{n,\omega} \mathbb{1}_{A_n} \circ F_{\alpha,\omega}(1) \cdot \mathbb{1}_{A_n}(1) = \sum_{\omega \in \Sigma} c_{1,\omega} \mathbb{1}_{A_n} \circ F_{\alpha,\omega}(1) = 0.$$

This completes the proof. \square

We will now show that conditions (H1) and (H2) are satisfied for every δ -expansive α -Farey system and for the Banach space $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$.

Proposition 2.4. *The pair $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$ forms a Banach space and for a δ -expansive α -Farey system, the operators R_n and $R(1)$ map \mathcal{B}_α into itself. Moreover, (H1) and (H2) are satisfied.*

In the proof of the above proposition we will make use of the following lemma.

Lemma 2.5. *For any α -Farey map F_α , we have that $c_{1,10_{n-1}} = \mu_\alpha(\{\phi = n\}) = a_n = t_n - t_{n+1}$, where $10_n := (1, \underbrace{0, 0, \dots, 0}_{n\text{-times}})$, for each $n \in \mathbb{N}$.*

Proof. By construction of the α -Farey map F_α , we have that $\{\phi = 1\} = [1 - a_1 t_1, 1 - a_1 t_2]$ and that $\{\phi = n\} = (1 - a_1 t_n, 1 - a_1 t_{n+1}]$, for all integers $n > 1$. Thus,

$$\mu_\alpha(\{\phi = n\}) = \int \mathbb{1}_{\{\phi=n\}} \cdot \frac{d\mu_\alpha}{d\lambda} d\lambda = a_1^{-1} \lambda(\{\phi = n\}) = t_n - t_{n+1}. \quad (15)$$

We will now show by induction on n that, for each $k \in \mathbb{N}$,

$$c_{k,10_{n-1}} = \frac{t_{k+n-1} - t_{k+n}}{t_k}. \quad (16)$$

From (2), we have that $c_{k,(1)} = 1 - t_{k+1}/t_k = (t_k - t_{k+1})/t_k$, for each $k \in \mathbb{N}$. Suppose that the statement in (16) is true for some $n \in \mathbb{N}$. From (13), we have that

$$c_{k,10_n} := (t_{k+1}/t_k) c_{k+1,10_{n-1}},$$

for each $k \in \mathbb{N}$, which gives

$$c_{k,10_n} = \frac{t_{k+1}}{t_k} \frac{t_{k+n} - t_{k+n+1}}{t_{k+1}} = \frac{t_{k+n} - t_{k+n+1}}{t_k}.$$

This completes the proof of the statement in (16).

Setting $k = 1$ in (16), we obtain that $c_{1,10_{n-1}} = t_n - t_{n+1}$, for all $n \in \mathbb{N}$. Combining this with (15), completes the proof. \square

Proof of Proposition 2.4. It is shown in [2, Section 1] that the pair $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$ forms a Banach space.

We now prove that condition (H1) holds and the invariance of \mathcal{B}_α . For this, let $u \in \mathcal{B}_\alpha$ and fix $k \in \mathbb{N}$. Applying Lemmas 2.2 and 2.5 we have that

$$R_k(u) = \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^k(\mathbb{1}_{\{\phi=k\}} \cdot u) = \mathbb{1}_{A_1} \cdot \mu_\alpha(\{\phi = k\}) \cdot u \circ F_{\alpha,10_{k-1}}.$$

Hence, by definition of the partition β_α , we have that $\|R_k(u)\|_{\mathcal{B}_\alpha} \leq \mu_\alpha(\{\phi = k\}) \|u\|_{\mathcal{B}_\alpha}$, and so, the operator R_k maps \mathcal{B}_α into itself. Further, by definition of $R(1)$, this gives that

$$\|R(1)u\|_{\mathcal{B}_\alpha} \leq \sum_{n \in \mathbb{N}} \|R_n(u)\|_{\mathcal{B}_\alpha} \leq \sum_{n \in \mathbb{N}} \mu_\alpha(\{\phi = n\}) \|u\|_{\mathcal{B}_\alpha} = \|u\|_{\mathcal{B}_\alpha},$$

and so, the operator $R(1)$ maps \mathcal{B}_α into itself. Linearity of R_k and $R(1)$ follows from the linearity of \widehat{F}_α .

For the proof of property (H2)(i), observe that G_α is a piecewise linear expansive map with the following properties.

- (i) On the set $\{\phi = n\}$, the absolute value of the derivative of G_α is equal to $1/(t_n - t_{n+1})$. Moreover, since $(t_n)_{n \in \mathbb{N}}$ is a positive monotonically decreasing sequence which is bounded above by 1, it follows that there exists a constant $c > 1$ with $1/(t_n - t_{n+1}) > c$, for all $n \in \mathbb{N}$.
- (ii) The partition β_α is a countable-infinite partition of A_1 and $G_\alpha(\{\phi = 1\}) = \overline{A_1}$ and $G_\alpha(\{\phi = n\}) = A_1$ if $n \geq 2$, and hence, $\mu_\alpha(G_\alpha(\{\phi = n\})) = \mu_\alpha(A_1) = 1$, for all $n \in \mathbb{N}$. Moreover, the σ -algebra generated by $\{G_\alpha^{-n}(\{\phi = m\}) : n, m \in \mathbb{N}\}$ is equal to the Borel σ -algebra on $\overline{A_1}$.
- (iii) For each $n \in \mathbb{N}$ and $\psi := (1, \underbrace{0, 0, \dots, 0}_{(n-1)\text{-times}}, 1)$, we have that

$$F_{\alpha, \psi}([0, 1]) = \overline{\{\phi = n\}} \quad \text{and} \quad \frac{d\mu_\alpha \circ F_{\alpha, \psi}}{d\mu_\alpha} = t_n - t_{n+1} = a_n.$$

Given these properties, (H2)(i) is a consequence of [2, Theorem 1.6]: the proof of which is based on the *Theorem on the difference of two norms* by Ionescu-Tulcea and Marinescu [13].

For the proof of property (H2)(ii), we distinguish between the cases $z \in \mathbb{D}$ and $z \in \mathbb{S}^1 \setminus \{1\}$.

Case 1. ($z \in \mathbb{D}$): Sarig showed in [21, Section 3] that

$$T(z) \circ R(z)u = (T(z)u) - u, \tag{17}$$

where the operators $T(z), T_n : \mathcal{L}_{\mu_\alpha}^1([0, 1]) \rightarrow \mathcal{L}_{\mu_\alpha}^1([0, 1])$ are defined by

$$T(z) := \sum_{n \in \mathbb{N}_0} z^n \cdot T_n \quad \text{and} \quad T_n(u) := \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^n(\mathbb{1}_{A_1} \cdot u).$$

By way of contradiction, suppose that 1 is an eigenvalue of $R(z)$ restricted to \mathcal{B}_α . Then there exists a non-zero measurable function $w \in \mathcal{B}_\alpha$ such that $R(z)w = w$. Substituting this into (17) shows that w is equal to zero μ_α -almost everywhere, which gives a contradiction.

Case 2. ($z \in \mathbb{S}^1 \setminus \{1\}$): We will now show that 1 is not an eigenvalue of $R(z)$. (This part of the proof is based on the proof of [10, Lemma 6.7].) Since $z \in \mathbb{S}^1 \setminus \{1\}$, there exists a $t \in (0, 2\pi)$ such that $z = e^{it}$. Suppose that $R(z)f = f$, for some non-zero $f \in \mathcal{B}_\alpha$. Let $\mathcal{L}_{\mu_\alpha}^2(A_1)$ denote the class of complex-valued measurable functions f with domain $\overline{A_1}$ for which $|f|^2$ is μ_α -integrable, and let it be equipped with the standard $\mathcal{L}_{\mu_\alpha}^2$ -inner product,

$$\langle u_1, u_2 \rangle := \int u_1 \cdot \overline{u_2} d\mu_\alpha.$$

For each $u \in \mathcal{L}_{\mu_\alpha}^2(A_1)$ set $\|u\|_2 := \langle u, u \rangle^{1/2}$. Further, set $\mathcal{L}^\infty(A_1) := \{v : \overline{A_1} \rightarrow \mathbb{C} : \|v\|_\infty < +\infty\}$ and define $V : \mathcal{L}^\infty \rightarrow \mathcal{L}^\infty$ by $V(u) := e^{-it\phi} \cdot u \circ G_\alpha$. Noting that $R(z)v = R(1)(e^{it\phi} \cdot v)$ and using (11), we have that, for all $v \in \mathcal{B}_\alpha$ and $u \in \mathcal{L}^\infty(A_1)$,

$$\langle u, R(z)v \rangle = \int \overline{u} \cdot R(z)v d\mu_\alpha = \int \overline{u} \cdot R(1)(e^{it\phi} \cdot v) d\mu_\alpha = \int \overline{u} \circ G_\alpha \cdot e^{it\phi} \cdot v d\mu_\alpha = \langle V(u), v \rangle.$$

Further,

$$\begin{aligned} \|V(f) - f\|_2^2 &= \|V(f)\|_2^2 - 2\Re\langle V(f), f \rangle + \|f\|_2^2 \\ &= \|V(f)\|_2^2 - 2\Re\langle f, R(z)(f) \rangle + \|f\|_2^2 \\ &= \|V(f)\|_2^2 - 2\Re\langle f, f \rangle + \|f\|_2^2 \\ &= \|V(f)\|_2^2 - \|f\|_2^2 \end{aligned} \tag{18}$$

and, as G_α preserves the measure μ_α restricted to \bar{A}_1 , we have that

$$\|V(f)\|_2^2 = \int |f|^2 \circ G_\alpha d\mu_\alpha = \int |f|^2 d\mu_\alpha = \|f\|_2^2. \quad (19)$$

Combining (18) and (19), it follows that $V(f) - f$ vanishes μ_α -almost everywhere on \bar{A}_1 . Thus, by taking the modulus, the ergodicity of G_α implies that $|f|$ is equal to a constant, μ_α -almost everywhere on \bar{A}_1 . As f does not vanish μ_α -almost everywhere, this constant is non-zero, and so, we obtain that $e^{-it\phi} = f/(f \circ G_\alpha)$ almost everywhere on \bar{A}_1 . Now, for each $n \in \mathbb{N}$, let $I_n \subseteq \{\phi = n\}$ be the interval of positive measure, such that $G_\alpha(I_n) = \{\phi = n\}$ and let

$$J_n := \{x \in I_n : G_\alpha(x) \notin I_n \text{ and } e^{it\phi(x)} = f(x)/f \circ G_\alpha(x)\}.$$

Since $e^{it\phi} = f/f \circ G_\alpha$ almost everywhere, and since the map G_α is linear and expanding, we have that $\mu_\alpha(J_n) > 0$. In particular, the set J_n is non-empty. We claim that there exists $y_n \in J_n$ such that $f(y_n) = f \circ G_\alpha(y_n)$, for each $n \in \mathbb{N}$. By way of contradiction, suppose that $f(x) \neq f \circ G_\alpha(x)$, for all $x \in J_n$. Since f is constant almost everywhere on \bar{A}_1 , we have that f is constant almost everywhere on $J_n \cup G_\alpha(J_n)$, which gives an immediate contradiction to the assumption that $f(x) \neq f \circ G_\alpha(x)$, for all $x \in J_n$. Therefore, we have that there exists $y_n \in \bar{A}_1$, such that $\phi(y_n) = n$, $f(y_n) = f \circ G_\alpha(y_n)$ and $e^{it\phi(y_n)} = f(y_n)/f \circ G_\alpha(y_n)$, for each $n \in \mathbb{N}$. Hence, we have that $e^{itin} = 1$, for all $n \in \mathbb{N}$, contradicting the initial choice of t . \square

Finally, in preparation for the proof of Theorem 1.3, let us make note of the following well known properties of slowly varying functions.

Lemma 2.6. *Let $L: [a, +\infty] \rightarrow \mathbb{R}$ be a positive slowly varying function, for some $a \in \mathbb{N}_0$.*

(i) [23, p. 2] *For a compact interval $I \subset \mathbb{R}^+$ we have that*

$$\lim_{x \rightarrow +\infty} L(px)/L(x) = 1$$

holds uniformly with respect to $p \in I$, and hence, for a fixed $b \in \mathbb{R}^+$,

$$\lim_{x \rightarrow +\infty} L(x-b)/L(x) = 1.$$

(ii) [23, p. 18] *For a fixed $b \in \mathbb{R}^+$ we have that*

$$\lim_{x \rightarrow +\infty} L(x)x^{-b} = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} L(x)x^b = +\infty.$$

(iii) [23, p. 41] *If L is continuous, strictly increasing and*

$$\lim_{x \rightarrow +\infty} L(x) = +\infty,$$

then, for a fixed $c \in (0, 1)$,

$$\lim_{x \rightarrow +\infty} L^{-1}(cx)/L^{-1}(x) = 0.$$

(iv) [23, p. 50] *If $M: [a+1, +\infty) \rightarrow \mathbb{R}$ is defined to be the linear interpolation of the function*

$$n \mapsto \sum_{k=a+1}^n L(k)k^{-1},$$

then M is a slowly varying function and

$$\lim_{x \rightarrow \infty} \frac{L(x)}{M(x)} = 0.$$

3. EXTENDING CONVERGENCE: PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Let us first recall that, for $x \in (0, 1]$ and $n \in \mathbb{N}$,

$$\begin{aligned} (\mathcal{P}_\alpha^{n+1}(h_\alpha \cdot v))(x) &= \mathcal{P}_\alpha((\mathcal{P}_\alpha^n(h_\alpha \cdot v))(x)) \\ &= (\mathcal{P}_\alpha^n(h_\alpha \cdot v))(F_{\alpha,0}(x)) \cdot |F'_{\alpha,0}(x)| + (\mathcal{P}_\alpha^n(h_\alpha \cdot v))(F_{\alpha,1}(x)) \cdot |F'_{\alpha,1}(x)|, \end{aligned}$$

which gives

$$(\mathcal{P}_\alpha^n(h_\alpha \cdot v))(F_{\alpha,0}(x)) = \frac{(\mathcal{P}_\alpha^{n+1}(h_\alpha \cdot v))(x) - (\mathcal{P}_\alpha^n(h_\alpha \cdot v))(F_{\alpha,1}(x)) \cdot |F'_{\alpha,1}(x)|}{|F'_{\alpha,0}(x)|}. \quad (20)$$

We proceed by induction as follows. The start of the induction is given by the assumption in the theorem. For the inductive step, assume that the statement holds for $\bigcup_{i=1}^k A_i$, for some $k \in \mathbb{N}$. Then consider some arbitrary $y \in A_{k+1}$, and let x denote the unique element in A_k such that $F_{\alpha,0}(x) = y$. Using (20), the fact that $\widehat{F}_\alpha = h_\alpha^{-1} \mathcal{P}_\alpha(h_\alpha \cdot v)$ and the inductive hypothesis in tandem with the assumption that $\lim w_n/w_{n+1} = 1$, we obtain that

$$\begin{aligned} w_n(\widehat{F}_\alpha^n(v))(y) &= w_n(\widehat{F}_\alpha^n(v))(F_{\alpha,0}(x)) = \frac{w_n(\mathcal{P}_\alpha^n(h_\alpha \cdot v))(F_{\alpha,0}(x))}{h_\alpha(F_{\alpha,0}(x))} \\ &= \frac{w_n(\mathcal{P}_\alpha^{n+1}(h_\alpha \cdot v))(x) - |F'_{\alpha,1}(x)| \cdot w_n(\mathcal{P}_\alpha^n(h_\alpha \cdot v))(F_{\alpha,1}(x))}{h_\alpha(F_{\alpha,0}(x)) \cdot |F'_{\alpha,0}(x)|} \\ &\sim \frac{h_\alpha(x) - h_\alpha(F_{\alpha,1}(x)) \cdot |F'_{\alpha,1}(x)|}{h_\alpha(F_{\alpha,0}(x)) \cdot |F'_{\alpha,0}(x)|} \Gamma_\delta \int v d\mu_\alpha = \Gamma_\delta \int v d\mu_\alpha, \end{aligned}$$

where the last equality is a consequence of the eigenequation $\mathcal{P}_\alpha h_\alpha = h_\alpha$. \square

Remark 3.1. Using an analogous proof to that given above, one can obtain that the result of Theorem 1.1 holds for other interval maps, such as Gibbs-Markov maps, Thaler maps and Pomeau-Manneville maps.

4. PROOF OF THEOREM 1.3

4.1. Asymptotics of the α -Farey transfer operator for $\delta = 1$. Throughout this section, we let $([0, 1], \mathcal{B}, \mu_\alpha, F_\alpha)$ be a 1-expansive α -Farey system. In order to prove Theorem 1.3 (i), we will use the following auxiliary results (Lemmas 4.1 and 4.2). Before which we require the following notation. Define the function $\ell : [0, +\infty) \rightarrow \mathbb{R}$ by

$$\ell(x) := \begin{cases} x/2 + 1/2 & \text{if } x \in [0, 1], \\ t_{n+1}(x - n) + w_n & \text{if } x \in [n, n+1], \text{ for } n \in \mathbb{N}. \end{cases} \quad (21)$$

Note that ℓ is the linear interpolation of the function $n \mapsto w_n$ defined on \mathbb{N}_0 , where $w_0 := 1/2$. Further, for $\sigma \in \mathbb{R}^+$, define

$$j_\sigma(x) := x - \ell^{-1}((1 + \sigma)^{-1} \ell(x)),$$

for all $x \geq \ell^{-1}((1 + \sigma)/2)$.

Lemma 4.1. *For a given $\sigma \in \mathbb{R}^+$, we have that $j_\sigma(x) \sim x$.*

Proof. For $\sigma \in \mathbb{R}^+$, we have that

$$\lim_{x \rightarrow \infty} \frac{j_\sigma(x)}{x} = 1 - \lim_{x \rightarrow \infty} \frac{\ell^{-1}(\ell(x)(1 + \sigma)^{-1})}{x} = 1 - \lim_{x \rightarrow \infty} \frac{\ell^{-1}(\ell(x)(1 + \sigma)^{-1})}{\ell^{-1}(\ell(x))} = 1,$$

where the last equality follows from the fact that ℓ is a positive, strictly monotonically increasing function and Lemma 2.6 (iii). \square

Here and in the sequel we will use the following notation. For $r \in \mathbb{R}$ we let $\lfloor r \rfloor$ denote the largest integer not exceeding r .

Lemma 4.2. *Let $(\delta_j)_{j \in \mathbb{N}}$ denote a sequence of positive real numbers such that $\sum_{j=1}^{\infty} \delta_j t_j < \infty$. If the wandering rate is moderately increasing, then*

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{w_n}{w_{n-j}} \delta_{j+1} t_{j+1} = \sum_{j=1}^{\infty} \delta_j t_j.$$

Proof. Without loss of generality, assume that $\sup\{\delta_j : j \in \mathbb{N}\} = 1$ and let $\sigma \in \mathbb{R}^+$ be fixed. By definition of ℓ , we have, for $n \geq \ell^{-1}((1+\sigma)/2)$, that

$$\begin{aligned} & \sum_{j=0}^n \frac{w_n}{w_{n-j}} \delta_{j+1} t_{j+1} \\ & \leq \frac{\ell(n)}{\ell(n-j_{\sigma}(n))} \sum_{j=0}^{\lfloor j_{\sigma}(n) \rfloor} \delta_{j+1} t_{j+1} + \frac{\ell(n)}{\ell(n(\ell(n))^{-2})} \sum_{j=\lfloor j_{\sigma}(n) \rfloor+1}^{\lfloor n-n(\ell(n))^{-2} \rfloor} \delta_{j+1} t_{j+1} + 2\ell(n) \sum_{j=\lfloor n-n(\ell(n))^{-2} \rfloor+1}^n t_{j+1}. \end{aligned}$$

By definition of $j_{\sigma}(n)$, we have that $\ell(n)/\ell(n-j_{\sigma}(n)) = (1+\sigma)^{-1}$. Further, by Lemma 2.6 (iv) and since $(t_j)_{j \in \mathbb{N}}$ is regularly varying sequence of order -1 , we have that,

$$\lim_{n \rightarrow \infty} 2\ell(n) \sum_{j=\lfloor n-n(\ell(n))^{-2} \rfloor+1}^n t_{j+1} \leq \lim_{n \rightarrow \infty} \frac{2t_{n+1}n}{\ell(n)} = 0.$$

If $\lfloor j_{\sigma}(n) \rfloor + 1 > \lfloor n-n(\ell(n))^{-2} \rfloor$, then this completes the proof. Otherwise, note that, by Lemma 2.6 (ii), we have that ℓ is a slowly varying function. Also, since ℓ is an unbounded monotonically increasing function we have that $n-n(\ell(n))^{-2} \sim n$ and, by Lemma 4.1, we have that $j_{\sigma}(n) \sim n$. The above three statements in tandem with the assumptions that $\sum_{j=1}^{\infty} \delta_j t_j < \infty$ and that the wandering rate is moderately increasing, yield the following:

$$\lim_{n \rightarrow \infty} \frac{\ell(n)}{\ell(n(\ell(n))^{-2})} \sum_{j=\lfloor j_{\sigma}(n) \rfloor+1}^{\lfloor n-n(\ell(n))^{-2} \rfloor} \delta_{j+1} t_{j+1} = 0.$$

This completes the proof in the case $\lfloor j_{\sigma}(n) \rfloor + 1 \leq \lfloor n-n(\ell(n))^{-2} \rfloor$. \square

Proof of Theorem 1.3 (i). By Theorem 2.1 and Proposition 2.4, we have for each $n \in \mathbb{N}$ that there exists $\theta_n : (0, 1) \rightarrow \mathbb{R}$ such that $\sup\{|\theta_n(x)| : x \in \bar{A}_1\} = o(1/w_n)$ and

$$\mathbb{1}_{\bar{A}_1} \cdot \widehat{F}_{\alpha}^n(\mathbb{1}_{\bar{A}_1} \cdot v) = \frac{1}{w_n} \int \mathbb{1}_{\bar{A}_1} \cdot v d\mu_{\alpha} \cdot \mathbb{1}_{\bar{A}_1} + \theta_n \cdot v \cdot \mathbb{1}_{\bar{A}_1}. \quad (22)$$

Set $\tau_{j,n} := w_n/w_{n-j} - 1$, for $n \in \mathbb{N}$ and $j \in \{0, 1, 2, \dots, n\}$. By (22), we have on \bar{A}_1 that

$$\begin{aligned} & \left| w_n \widehat{F}_{\alpha}^n(v) - \int v d\mu_{\alpha} \right| \\ & = \left| w_n \sum_{j=0}^n \mathbb{1}_{A_1} \cdot \widehat{F}_{\alpha}^{n-j}(\mathbb{1}_{A_1} \cdot \widehat{F}_{\alpha}^j(v \cdot \mathbb{1}_{A_{j+1}})) - \int v d\mu_{\alpha} \right| \\ & = \left| w_n \sum_{j=0}^n \frac{1}{w_{n-j}} \int \widehat{F}_{\alpha}^j(v \cdot \mathbb{1}_{A_{j+1}}) d\mu_{\alpha} - \int v d\mu_{\alpha} + w_n \sum_{j=0}^n \theta_{n-j} \cdot \widehat{F}_{\alpha}^j(v \cdot \mathbb{1}_{A_{j+1}}) \right| \\ & \leq \sum_{j=0}^n \tau_{n,j} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\mu_{\alpha} + w_n \sum_{j=0}^n \|\theta_{n-j}\|_{\infty} \|\widehat{F}_{\alpha}^j(v \cdot \mathbb{1}_{A_{j+1}})\|_{\infty} + \sum_{j=n+1}^{\infty} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\mu_{\alpha}. \end{aligned} \quad (23)$$

Since $v \in \mathcal{A}_{\alpha} \subseteq \mathcal{L}_{\mu_{\alpha}}^1([0, 1])$, it follows that in the final line of (23) the third term converges to zero. To see that the first and the second term in the final line of (23) converge to zero, observe that

(i) since $v \in \mathcal{A}_\alpha$, we have that $v \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ and, moreover,

$$\int |v \cdot \mathbb{1}_{A_j}| d\mu_\alpha = \frac{t_j}{a_j} \int v \cdot \mathbb{1}_{A_j} d\lambda;$$

(ii) since $v \in \mathcal{A}_\alpha$ we have that $\|v\|_\infty$ is finite, and so the sequence

$$\left(\frac{1}{a_j} \int v \cdot \mathbb{1}_{A_j} d\lambda \right)_{j \in \mathbb{N}}$$

is a bounded sequence;

(iii) using Lemma 2.3 together with the fact that \widehat{F}_α is positive and linear and the fact that if $v \in \mathcal{A}_\alpha$, then $\|v\|_\infty$ is finite, we have that $|\widehat{F}_\alpha^{j-1}(v \cdot \mathbb{1}_{A_j})(x)| \leq \|v\|_\infty t_j$;

(iv) given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\|\theta_m\|_\infty \leq \epsilon/\ell(m)$, for all $m \geq N_\epsilon$.

Combining these observations with Lemma 4.2 and (4), we have that the first and the second term in the final line of (23) converge to zero. Since the arguments given above are independent of a given point in \overline{A}_1 , an application of Theorem 1.1 now finishes the proof. \square

Remark 4.3. In the proof of Theorem 1.3 (i) we have not used the specific structure of \mathcal{B}_α . We only used that \mathcal{B}_α is a Banach space which satisfies conditions (H1) and (H2). Thus, we may replace \mathcal{B}_α by an arbitrary Banach space which satisfies conditions (H1) and (H2). For such alternative Banach spaces see Remark 1.8. In doing such a substitution one may change the uniform convergence to almost everywhere uniform convergence.

To conclude, we give examples of 1-expansive α -Farey systems and of observables which belong to the set \mathcal{A}_α and which satisfy the summability condition given in (4).

Example 4.4. Let $([0, 1], \mathcal{B}, \mu_\alpha, F_\alpha)$ denote a 1-expansive α -Farey system with moderately increasing wandering rate. Set $u := f/h_\alpha$, where $f \in \mathcal{D}_{\mu_\alpha}$, for

$$\mathcal{D}_{\mu_\alpha} := \{f : f \in \mathcal{L}_{\mu_\alpha}^1([0, 1]) \text{ and } f \in C^2((0, 1)) \text{ with } f' > 0 \text{ and } f'' \leq 0\}.$$

We claim that $u \in \mathcal{A}_\alpha$ and moreover, that u satisfies the summability condition given in (4).

We first verify that $u \in \mathcal{A}_\alpha$. For this, we are required to show that $u \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$, that $\|u\|_\infty < +\infty$ and that $\widehat{F}_\alpha^{j-1}(u \cdot \mathbb{1}_{A_j}) \in \mathcal{B}_\alpha$, for all $j \in \mathbb{N}$. By definition, any function belonging to \mathcal{D}_{μ_α} is convex and continuous on $(0, 1)$, twice differentiable and μ_α -integrable. Thus, $u \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ and $\|u\|_\infty < +\infty$. Combining this with the fact that $1/h_\alpha$ is μ_α integrable, non-negative and bounded, we have that $u \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ and $\|u\|_\infty < +\infty$. Let us now turn to the second assertion, namely that $\widehat{F}_\alpha^{n-1}(u \cdot \mathbb{1}_{A_n}) \in \mathcal{B}_\alpha$, for all $n \in \mathbb{N}$. We immediately have that $\widehat{F}_\alpha^0(u \cdot \mathbb{1}_{A_1}) = u \cdot \mathbb{1}_{A_1} \in \mathcal{B}_\alpha$. For $n \geq 2$, note that, if g is a differentiable Lipschitz function on \overline{A}_1 , then $D_\alpha(g) = \sup\{|g'| : x \in \overline{A}_1\}$. Thus, by Lemma 2.3 and the chain rule, we have that, for each integer $n \geq 2$,

$$\begin{aligned} \|\widehat{F}_\alpha^{n-1}(u \cdot \mathbb{1}_{A_n})\|_{\mathcal{B}_\alpha} &= \|c_{1,0_{n-1}}((f/h_\alpha) \circ F_{\alpha,0_{n-1}})\|_{\mathcal{B}_\alpha} \\ &= \|a_n f \circ F_{\alpha,0_{n-1}}\|_{\mathcal{B}_\alpha} \\ &= \|a_n f \circ F_{\alpha,0_{n-1}}\|_\infty + D_\alpha(a_n f \circ F_{\alpha,0_{n-1}}) \\ &\leq a_n(\|f\|_\infty + \|f' \cdot \mathbb{1}_{A_n}\|_\infty). \end{aligned} \tag{24}$$

Since $f \in \mathcal{D}_{\mu_\alpha}$, we have that $\|f\|_\infty < +\infty$ and that $0 \leq f'(x) \leq (f(t_{n+1}) - f(t_{n+2}))/a_{n+1}$, for all $x \in A_n$. Therefore, since $a_n = l(n)n^{-2}$, it follows that there exists $c > 0$, such that

$$\sum_{n \in \mathbb{N}, n \neq 1} a_n \|f' \cdot \mathbb{1}_{A_n}\|_\infty \leq \sum_{n \in \mathbb{N}, n \neq 1} a_n f'(\xi_{n+1}) = \sum_{n \in \mathbb{N}, n \neq 1} \frac{a_n}{a_{n+1}} (f(t_{n+1}) - f(t_{n+2})) \leq c f(t_3).$$

Combining this with (24) and using the facts that the sequence $(a_n)_{n \in \mathbb{N}}$ is summable and that $\|f\|_\infty$ and $\|u \cdot \mathbb{1}_{A_1}\|_{\mathcal{B}_\alpha}$ are finite, the summability condition in (4) follows. Hence it follows that $u \in \mathcal{A}_\alpha$.

4.2. Asymptotics of the α -Farey transfer operator for $\delta \in (1/2, 1]$.

Proof of Theorem 1.3 (ii). Recall that $([0, 1], \mathcal{B}, \mu_\alpha, F_\alpha)$ is a δ -expansive α -Farey system and that $\Gamma_\delta = (\Gamma(1 + \delta)\Gamma(2 - \delta))^{-1}$. From (1) and (14), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \|\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}/h_\alpha)\|_{\mathcal{B}_\alpha} &= \|\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}/h_\alpha)\|_\infty + D_\alpha(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}/h_\alpha)) \\ &= a_n \|v \cdot \mathbb{1}_{A_n}\|_\infty + a_n D_\alpha(v \cdot \mathbb{1}_{A_n} \circ F_{\alpha, 0_{n-1}}). \end{aligned}$$

Combining this with the assumptions of the theorem, there exists a constant $c' > 0$ such that $\|\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}/h_\alpha)\|_{\mathcal{B}_\alpha} < c'$, for all $n \in \mathbb{N}$.

As in the proof of Theorem 1.3 (i) we have, by Theorem 2.1 and Proposition 2.4, that there exists $\theta_n : [0, 1] \rightarrow \mathbb{R}$ such that $\sup\{|\theta_n(x)| : x \in \overline{A_1}\} = o(1/w_n)$ and, for each $n \in \mathbb{N}_0$,

$$\mathbb{1}_{\overline{A_1}} \cdot \widehat{F}_\alpha^n(\mathbb{1}_{\overline{A_1}} \cdot v/h_\alpha) = \frac{\Gamma_\delta}{w_n} \int \mathbb{1}_{\overline{A_1}} \cdot v d\lambda \mathbb{1}_{\overline{A_1}} + \theta_n \cdot v/h_\alpha \cdot \mathbb{1}_{\overline{A_1}}.$$

Set $\tau_{j,n} := w_n/w_{n-j} - 1$, for each $n \in \mathbb{N}$ and $j \in \{0, 1, 2, \dots, n\}$. By a calculation similar as in (23), we have on $\overline{A_1}$ that

$$\begin{aligned} \left| w_n \widehat{F}_\alpha^n(v/h_\alpha) - \Gamma_\delta \int v d\lambda \right| &\leq w_n \sum_{j=0}^n \|\theta_{n-j}\|_\infty \|\widehat{F}_\alpha^j(v \cdot \mathbb{1}_{A_{j+1}}/h_\alpha)\|_\infty \\ &\quad + \Gamma_\delta \sum_{j=0}^n \tau_{n,j} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda + \Gamma_\delta \sum_{j=n+1}^\infty \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda. \end{aligned} \tag{25}$$

As $v \in \mathcal{L}_\lambda^1([0, 1])$, the third summand on the RHS of (25) tends to zero. To see that the second term on the RHS of (25) converges to zero, recall that the sequence $(t_n)_{n \in \mathbb{N}}$ is positive, monotonically decreasing and bounded above by one and that by assumption $a_n = \delta l(n)n^{-(1+\delta)}$. Further, by (12), given $\xi > 0$, there exist constants $c'' \in \mathbb{R}$ and $N = N(\xi) \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$ with $n \geq N$,

$$w_m \geq \bar{\Gamma}_\delta (e^{-\xi} l(m)m^{1-\delta} + c'') \quad \text{and} \quad \bar{\Gamma}_\delta e^\xi l(n)n^{1-\delta} \geq w_n \geq \bar{\Gamma}_\delta e^{-\xi} l(n)n^{1-\delta}.$$

This implies that given $\epsilon \in (0, 1)$, we have for all $n \in \mathbb{N}$ such that $\lceil \epsilon n \rceil > N$,

$$\begin{aligned} &\sum_{j=n-\lceil \epsilon n \rceil+1}^n \frac{w_n}{w_{n-j}} \frac{l(j)}{j^{1+\delta-\eta}} \\ &\leq \frac{\bar{\Gamma}_\delta e^\xi (l(n))^2}{w_0 n^{2\delta-\eta}} + \sum_{j=n-\lceil \epsilon n \rceil+1}^{n-N} \frac{e^{2\xi} n^{1-\delta} l(n)}{(n-j)^{1-\delta} l(n-j)} \frac{l(j)}{j^{1+\delta-\eta}} + \sum_{j=n-N+1}^{n-1} \frac{e^{2\xi} l(n)n^{1-\delta}}{(n-j)^{1-\delta} l(n-j) + c'' e^\xi} \frac{l(j)}{j^{1+\delta-\eta}}. \end{aligned}$$

(Recall that η is the value given in condition (b) of Theorem 1.3 (ii)). A simple calculation shows that the RHS of the latter inequality converges to zero as n tends to infinity. Further, since all of the summand are positive, we have that the limit as n tends to infinity exists and equals zero. This together with Lemma 2.6 (i) implies that, for each $\epsilon, \xi \in (0, 1)$,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \sum_{j=0}^n \frac{w_n}{w_{n-j}} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda &\leq \limsup_{n \rightarrow +\infty} \sum_{j=0}^{n-\lceil \epsilon n \rceil} \frac{e^{2\xi} l(n)n^{1-\delta}}{l(n-j)(n-j)^{1-\delta}} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda \\ &= \limsup_{n \rightarrow +\infty} \sum_{j=0}^{n-\lceil \epsilon n \rceil} \frac{e^{2\xi} n^{1-\delta}}{(n-j)^{1-\delta}} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda \\ &\leq \epsilon^{\delta-1} e^{2\xi} \int |v| d\lambda. \end{aligned} \tag{26}$$

Furthermore,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \sum_{j=0}^n \frac{w_n}{w_{n-j}} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda &\geq \liminf_{n \rightarrow +\infty} \sum_{j=0}^{n-\lceil \epsilon n \rceil} \frac{e^{-2\xi} l(n) n^{1-\delta}}{l(n-j)(n-j)^{1-\delta}} \int |v \cdot \mathbb{1}_{A_{j+1}}| d\lambda \\ &\geq \epsilon^{\delta-1} e^{-2\xi} \int |v| d\lambda. \end{aligned}$$

Since $\epsilon, \xi \in (0, 1)$ are arbitrary, it follows that

$$\lim_{n \rightarrow +\infty} \sum_{j=1}^n \frac{w_n}{w_{n-j}} \int |v \cdot \mathbb{1}_{A_j}| d\lambda = \int |v| d\lambda,$$

and hence, the second term on the RHS of (25) converges to zero as n tends to infinity. We now show that the first term on the RHS of (25) converges to zero. By (12) and the fact that $\sup\{|\theta_n(x)| : x \in \bar{A}_1\} = o(1/w_n)$, given $\xi > 0$ there exists $M = M(\xi) \in \mathbb{N}$ such that, for all $m \geq M$,

$$\bar{\Gamma}_\delta e^{-\xi} l(m) m^{1-\delta} \leq w_m \leq \bar{\Gamma}_\delta e^\xi l(m) m^{1-\delta} \quad \text{and} \quad \|\theta_m\|_\infty \leq \xi/w_m.$$

Moreover, there exists a constants $c_1, c_2 \in \mathbb{R}$ such that, for all $n \in \mathbb{N}_0$,

$$w_n \geq \bar{\Gamma}_\delta (e^{-\xi} l(n) n^{1-\delta} + c_1) \quad \text{and} \quad \|\theta_n\|_\infty \leq c_2/w_n.$$

Furthermore, since F_α is δ -expansive, by condition (b) in Theorem 1.3 (ii), we have that the sequence $(a_n \|v \cdot \mathbb{1}_{A_n}\|_\infty)_{n \in \mathbb{N}}$ is summable. These properties together with Lemma 2.3 and an argument similar to that presented in (26), imply the existence of a constant $c_3 > 0$ such that

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow +\infty} w_n \sum_{j=0}^n \|\theta_{n-j}\|_\infty \|\widehat{F}_\alpha^j(v \cdot \mathbb{1}_{A_{j+1}}/h_\alpha)\|_\infty \\ &\leq \limsup_{n \rightarrow +\infty} \xi e^{2\xi} \sum_{j=0}^{n-M} \frac{l(n) n^{1-\delta}}{l(n-j)(n-j)^{1-\delta}} \|v \cdot \mathbb{1}_{A_{j+1}}\|_\infty a_{j+1} + \limsup_{n \rightarrow +\infty} \frac{c_2 w_n a_{n+1}}{w_0} \\ &\quad + \limsup_{n \rightarrow +\infty} c_3 e^{2\xi} \sum_{j=n-M+1}^{n-1} \frac{l(n) n^{1-\delta}}{(n-j)^{1-\delta} l(n-j) + c_1 e^\xi} a_{j+1} \|v \cdot \mathbb{1}_{A_{j+1}}\|_\infty \\ &= \xi e^{2\xi} \sum_{j=0}^\infty a_{j+1} \|v \cdot \mathbb{1}_{A_{j+1}}\|_\infty. \end{aligned}$$

Since $\xi > 0$ was chosen arbitrarily, the result follows and, since the arguments given above are independent of a given point in \bar{A}_1 , an application of Theorem 1.1 finishes the proof. \square

4.3. Proof of Theorem 1.3 (iii) - Counterexamples for $\delta \in (1/2, 1)$. In this section we provide a constructive proof of Theorem 1.3 (iii). The proof is divided into several parts. First, we define a class of observables \mathcal{V} . Second, in Proposition 4.9 we will show that if $v \in \mathcal{V}$, then v is bounded, of bounded variation, Riemann integrable and belongs to $\mathcal{L}_{\mu_\alpha}^1([0, 1])$. Third, in Proposition 4.10 we will show that if $v \in \mathcal{V}$, then it belongs to the space \mathcal{A}_α , and in Proposition 4.11 we will show that the summability condition given in (4) is satisfied for all $v \in \mathcal{V}$. Finally, in Proposition 4.13 we will show that, if $v \in \mathcal{V}$, then

$$\liminf_{n \rightarrow +\infty} w_n \widehat{F}_\alpha^n(v)(x) = \Gamma_\delta \int v d\mu_\alpha \quad \text{and} \quad \limsup_{n \rightarrow +\infty} w_n \widehat{F}_\alpha^n(v)(x) = +\infty.$$

Combing these results will then yield a proof of Theorem 1.3 (iii)

Let us now begin by defining the set \mathcal{V} . We let \mathcal{V} denote the class of observable $v : [0, 1] \rightarrow \mathbb{R}$ which are of the following form:

$$v := \sum_{k=1}^\infty s_k \sum_{j=N_k-n_k}^{N_k} \mathbb{1}_{A_j}, \quad (27)$$

where

$$N_k := \lceil 2^{g_1 k} \rceil, \quad n_k := \lfloor 2^{g_2 k} \rfloor \quad \text{and} \quad s_k := 2^{-g_3 k},$$

and where g_1, g_2 and g_3 denote three positive constants, depending on δ , such that

(C1): $g_1 > (1 - \delta)^{-1}$,

(C2): $\delta g_1 > g_2$,

(C3): there exists $\epsilon \in (0, \delta - 1/2)$, such that $g_2(\delta - \epsilon) > (2\delta + 2\epsilon - 1)g_1 + g_3$.

Example 4.5. For $\delta \in (1/2, 1)$, choose $g_1 = (1 + \epsilon)/(1 - \delta)$ and $g_2 = \delta/(1 - \delta)$. Then it is clear that g_1 and g_2 satisfy the conditions (C1) and (C2). With these choices one immediately verifies that (C3) is equivalent to $g_3 < (1 - \delta) - \rho\epsilon$, for $\rho := (3\delta + 1 + 2\epsilon)/(1 - \delta)$. Hence, by choosing $\epsilon > 0$ sufficiently small, it follows that the conditions (C1), (C2) and (C3) can be fulfilled simultaneously.

Example 4.6. For $\delta \in (1/2, 1)$, choose $g_1 = (1 - \delta)^{-2}$, $g_2 = (\delta^2 + 2\delta - 1)(2\delta)^{-1}(1 - \delta)^{-2}$, $g_3 = 8^{-1}$ and $\epsilon = \delta(1 - \delta)^2(2\delta^2 + 12\delta - 2)^{-1}$. For each $\delta \in (1/2, 1)$, these values are all positive and satisfy conditions (C1), (C2) and (C3).

The main reason why we require the sequence $(s_k)_{k \in \mathbb{N}}$ is to ensure that v is of bounded variation. Further, condition (C3) is only required in the proof of the second statement of Proposition 4.13, specifically when Lemma 4.12 is used.

Before we begin with Proposition 4.9, we give two elementary lemmas which we will use in its proof.

Lemma 4.7. *If $s \in (0, 1)$ and $1 < b < a^s$, then*

$$\sum_{k=1}^{\infty} a^{(1-s)k} - (a^k - b^k)^{1-s} \leq \sum_{k=1}^{\infty} (b/a^s)^k < +\infty.$$

Proof. By assumption, we have that $a/b > 1$ and so $1 - (b/a)^k \leq (1 - (b/a)^k)^{1-s}$, which implies that $(a/b)^k - (a/b)^k(1 - (b/a)^k)^{1-s} \leq 1$, for each $k \in \mathbb{N}$. Therefore,

$$\sum_{k=1}^{\infty} a^{(1-s)k} - (a^k - b^k)^{1-s} = \sum_{k=1}^{\infty} (b/a^s)^k ((a/b)^k - (a/b)^k(1 - (b/a)^k)^{1-s}) \leq \sum_{k=1}^{\infty} (b/a^s)^k < +\infty.$$

□

Lemma 4.8. *For each $k \in \mathbb{N}$, we have that $N_{k+1} - n_{k+1} > N_k$.*

Proof. By (C1) and (C2), we have that $g_1 > (1 - \delta)^{-1}$ and $g_2 - g_1 < 0$. This implies that, for all $k \in \mathbb{N}$,

$$\begin{aligned} \frac{N_{k+1} - n_{k+1}}{N_k} &\geq \frac{2^{g_1(k+1)} - 2^{g_2(k+1)} - 1}{2^{g_1 k} + 1} \\ &= \frac{2^{g_1}(1 - 2^{(g_2 - g_1)(k+1)} - 2^{-g_1(k+1)})}{2^{-g_1 k} + 1} \\ &\geq \frac{2^{g_1}(1 - 2^{2(g_2 - g_1)} - 2^{-2g_1})}{2^{-g_1} + 1}. \end{aligned}$$

Using (C1) and (C2) once more immediately verifies that the latter term is strictly greater than one. □

Proposition 4.9. *An observable v defined as in (27) is bounded, of bounded variation, Riemann integrable and belongs to the space $\mathcal{L}_{\mu_\alpha}^1([0, 1])$.*

Proof. Clearly the observable v is Riemann integrable. Moreover, v is measurable, as each of the atoms of α is measurable and v is the sum of indicator functions of atoms of α . Further, the range of v is equal to $\{0\} \cup \{s_k : k \in \mathbb{N}\}$, and thus, $\|v\|_\infty = s_1$. By Lemma 4.8, we have that $N_{k+1} - n_{k+1} > N_k$, and so the variation of v is equal to $2 \sum_{k=1}^{\infty} s_k = 1$, which is finite, as $s_k := 2^{-g_3 k}$ and as g_3 is positive. This shows that v is of bounded variation. It

remains to show that v is μ_α -integrable. For this recall that $\mu_\alpha(A_k) = t_k$, for each $k \in \mathbb{N}$. Choose a positive constant $\eta < \min\{\delta, g_3/g_1\}$ and recall that $t_n \sim l(n)n^{-\delta}$. By Lemma 2.6 (ii), there exists a constant $c > 0$ such that $t_n \leq cl(n)n^{-\delta} \leq cn^{\eta-\delta}$, for each $n \in \mathbb{N}$. Therefore, by Lemma 4.7 and Lemma 4.8, we have that

$$\begin{aligned} \int |v| d\mu_\alpha &= \sum_{k=1}^{\infty} s_k \sum_{j=N_k-n_k}^{N_k} t_j \\ &\leq \sum_{k=1}^{\infty} 2^{-g_3 k} \sum_{j=N_k-n_k}^{N_k} \frac{c}{j^{\delta-\eta}} \\ &\leq (1-\delta+\eta)^{-1} c \sum_{k=1}^{\infty} \left(2^{-g_3 k} \left(2^{g_1 k(1-\delta+\eta)} - (2^{g_1 k} - 2^{g_2 k})^{1-\delta+\eta} \right) + 2^{-g_3 k} N_k^{\eta-\delta} \right) \\ &\leq (1-\delta+\eta)^{-1} c \sum_{k=1}^{\infty} \left(2^{(g_2-\delta g_1+\eta g_1-g_3)k} + 2^{-g_3 k} N_k^{\eta-\delta} \right). \end{aligned}$$

The latter series converges, since $\eta < \min\{\delta, g_3/g_1\}$, $g_2 < \delta g_1$ and $N_k > 1$, for all $k \in \mathbb{N}$. \square

Our next aim is to show that v belongs to \mathcal{A}_α and satisfies the condition given in (4).

Proposition 4.10. *An observable v defined as in (27) belongs to \mathcal{A}_α .*

Proof. By Proposition 4.9, we have that $v \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$ and that $\|v\|_\infty = 1$. Moreover, by Lemma 2.3, we have on $[0, 1]$, that, for each $j \in \mathbb{N}$,

$$\widehat{F}_\alpha^{j-1}(v \cdot \mathbb{1}_{A_j})(x) = \begin{cases} t_j & \text{if } N_k - n_k \leq j \leq N_k \text{ for some } k \in \mathbb{N} \text{ and if } x \in A_1, \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Therefore, $\widehat{F}_\alpha^{j-1}(v \cdot \mathbb{1}_{A_j}) \in \mathcal{B}_\alpha$, for all $j \in \mathbb{N}$, and hence, it follows that $v \in \mathcal{A}_\alpha$. \square

Proposition 4.11. *An observable v defined as in (27) satisfies the summability condition given in (4).*

Proof. Lemma 2.3 and (28) together imply that

$$\sum_{k=0}^{\infty} \|\widehat{F}_\alpha^k(v \cdot \mathbb{1}_{A_{k+1}})\|_\infty = \sum_{k=1}^{\infty} s_k \sum_{j=N_k-n_k}^{N_k} t_j = \sum_{k=1}^{\infty} s_k \sum_{j=N_k-n_k}^{N_k} \mu_\alpha(A_j) = \int |v| d\mu_\alpha.$$

The latter term is finite, since $v \in \mathcal{L}_{\mu_\alpha}^1([0, 1])$, by Proposition 4.9. \square

In the proof of Proposition 4.13, we will require the following auxiliary result.

Lemma 4.12. *For each $N \in \mathbb{N}$, the following sequence diverges to infinity:*

$$\left(s_k \sum_{j=N_k-n_k}^{N_k-N} \frac{N_k^{1-\delta} l(N_k)}{(N_k-j)^{1-\delta} l(N_k-j)} \frac{l(j)}{j^\delta} \right)_{k \in \mathbb{N}}.$$

Proof. The result follows immediately from combining the following three observations.

- (i) Using the facts that $\delta \in (1/2, 1)$ and $\epsilon > 0$, that the sequence $(N_k)_{k \in \mathbb{N}}$ is not bounded above and is strictly monotonically increasing, that $s_k := 2^{-g_3 k}$ and that N is a fixed natural number, we have that

$$\lim_{k \rightarrow +\infty} s_k \left(\frac{N_k}{N_k - N} \right)^\delta N_k^{1-2\delta-2\epsilon} N^{\delta-\epsilon} = 0.$$

(ii) For each $k \in \mathbb{N}$, we have that

$$\begin{aligned} s_k N_k^{1-2\delta-2\epsilon} n_k^{\delta-\epsilon} &\geq 2^{-g_3 k} 2^{g_1(1-2\delta-2\epsilon)k} (2^{g_2(\delta-\epsilon)} - 1) \\ &= 2^{(g_1(1-2\delta-2\epsilon)+g_2(\delta-\epsilon)-g_3)k} - 2^{(g_1(1-2\delta-2\epsilon)-g_3)k}. \end{aligned}$$

Using condition (C3) with the facts that $\delta \in (1/2, 1)$, $\epsilon > 0$ and that g_1, g_2 and g_3 are positive, it follows that

$$\lim_{k \in \mathbb{N}} s_k N_k^{1-2\delta-2\epsilon} n_k^{\delta-\epsilon} = +\infty.$$

(iii) There exist constants $\kappa, \xi > 0$ such that, for all $k \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} &\sum_{j=N_k-n_k}^{N_k-N} \frac{N_k^{1-\delta} l(N_k)}{(N_k-j)^{1-\delta} l(N_k-j)} \frac{l(j)}{j^\delta} \\ &\geq \left(\frac{1}{N_k-N} \right)^\delta l(N_k-N) l(N_k) N_k^{1-\delta} e^{-\xi} \sum_{j=N_k-n_k}^{N_k-N} \frac{1}{(N_k-j)^{1-\delta} l(N_k-j)} \\ &\geq \kappa e^{-\xi} \left(\frac{N_k}{N_k-N} \right)^\delta N_k^{1-2\delta-2\epsilon} (n_k^{\delta-\epsilon} - N^{\delta-\epsilon}). \end{aligned}$$

Here, the first inequality follows from the facts that l is a slowly varying function and that $\lim_{n \rightarrow \infty} (N_k - n_k)/N_k = 1$ together with Lemma 2.6 (i). The second inequality follows from Lemma 2.6 (ii), which guarantees the existence of the constant $\kappa > 0$ such that $\kappa^{-1} n^\epsilon \geq l(n) \geq \kappa n^{-\epsilon}$, for all $n \in \mathbb{N}$.

□

Proposition 4.13. *For an observable v as defined in (27), we have that, on \bar{A}_1 ,*

$$\liminf_{n \rightarrow +\infty} w_n \widehat{F}_\alpha^n(v) = \Gamma_\delta \int v d\mu_\alpha \quad \text{and} \quad \limsup_{n \rightarrow +\infty} w_n \widehat{F}_\alpha^n(v) = +\infty. \quad (29)$$

Proof. By Theorem 2.1 and Proposition 2.4, we have uniformly on \bar{A}_1 that

$$\lim_{n \rightarrow +\infty} l(n) n^{1-\delta} \mathbb{1}_{\bar{A}_1} \cdot \widehat{F}_\alpha^n(\mathbb{1}_{A_1}) = (\Gamma_\delta / \bar{\Gamma}_\delta) \mu_\alpha(A_1) \mathbb{1}_{\bar{A}_1} = (\Gamma_\delta / \bar{\Gamma}_\delta) \mathbb{1}_{\bar{A}_1}.$$

Thus, given $\xi > 0$, there exists $N = N(\xi) \in \mathbb{N}$ such that, for all $n \geq N$ on \bar{A}_1 ,

$$\frac{e^\xi \Gamma_\delta n^{\delta-1}}{\bar{\Gamma}_\delta l(n)} \geq \widehat{F}_\alpha^n(\mathbb{1}_{A_1}) \geq \frac{e^{-\xi} \Gamma_\delta n^{\delta-1}}{\bar{\Gamma}_\delta l(n)}. \quad (30)$$

We will first show the second statement in (29). For this, observe that by (12) it is sufficient to show that, on A_1 ,

$$\limsup_{k \rightarrow +\infty} l(N_k) N_k^{1-\delta} \widehat{F}_\alpha^{N_k}(v)(x) = +\infty.$$

In order to see this, let $\xi > 0$ be fixed and let $p = p(\xi) \in \mathbb{N}$ denote the smallest integer for which $n_p > N$. Since \widehat{F}_α is a positive linear operator, we have, for all $k > p$, that

$$l(N_k) N_k^{1-\delta} \widehat{F}_\alpha^{N_k}(v) \geq s_k l(N_k) N_k^{1-\delta} \sum_{j=N_k-n_k}^{N_k-N} \widehat{F}_\alpha^{N_k}(\mathbb{1}_{A_j}). \quad (31)$$

Now, Lemma 2.6 (i) implies that $\lim_{n \rightarrow +\infty} (n^{1-\delta} l(n)) / ((n+1)^{1-\delta} l(n+1)) = 1$. As the sequence $(a_n)_{n \in \mathbb{N}}$ is positive and since $a_n = \delta n^{-1-\delta} l(n)$ the value $r := \inf\{(n^{1-\delta} l(n)) / ((n+1)^{1-\delta} l(n+1))\}$ is finite and strictly greater than zero. Hence, by (14), (30) and (31) and the fact that

$t_n \sim l(n)n^{-\delta}$, we have on \bar{A}_1 that, for each $k \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} l(N_k)N_k^{1-\delta}\widehat{F}_\alpha^{N_k}(v) &\geq e^{-\xi}\frac{\Gamma_\delta}{\bar{\Gamma}_\delta}s_k \sum_{j=N_k-n_k}^{N_k-N} \frac{(N_k-j)^{1-\delta}l(N_k-j)}{(N_k-j+1)^{1-\delta}l(N_k-j+1)} \frac{N_k^{1-\delta}l(N_k)}{(N_k-j)^{1-\delta}l(N_k-j)} t_j \\ &\geq e^{-2\xi}\frac{\Gamma_\delta}{\bar{\Gamma}_\delta}rs_k \sum_{j=N_k-n_k}^{N_k-N} \frac{N_k^{1-\delta}l(N_k)}{(N_k-j)^{1-\delta}l(N_k-j)} \frac{l(j)}{j^\delta}. \end{aligned}$$

By Lemma 4.12, the latter term diverges.

All that remains to show is that the first statement of (29) holds. For this, observe that, by positivity and linearity of \widehat{F} , Theorem 2.1, Proposition 2.4 and (14), we have on \bar{A}_1 that, for each $k \in \mathbb{N}$,

$$\begin{aligned} \Gamma_\delta \sum_{l=1}^{N_k} \int v \cdot \mathbb{1}_{A_l} d\mu_\alpha &= \Gamma_\delta \sum_{m=1}^k s_m \sum_{j=N_m-n_m}^{N_m} t_j = \sum_{m=1}^k s_m \sum_{j=N_m-n_m}^{N_m} \liminf_{n \rightarrow +\infty} w_n \widehat{F}^{n-j+1}(\widehat{F}_\alpha^{j+1}(\mathbb{1}_{A_j})) \\ &\leq \liminf_{n \rightarrow +\infty} w_n \widehat{F}^n \left(\sum_{m=1}^k s_m \sum_{j=N_m-n_m}^{N_m} \mathbb{1}_{A_j} \right) \\ &\leq \liminf_{n \rightarrow +\infty} w_n \widehat{F}^n(v). \end{aligned}$$

Since $k \in \mathbb{N}$ was arbitrary, the above inequalities imply that on \bar{A}_1 ,

$$\liminf_{n \rightarrow +\infty} w_n \widehat{F}^n(v) \geq \Gamma_\delta \int v d\mu_\alpha.$$

Suppose that the latter inequality is strict, namely, suppose that there exists a constant $c > 0$ such that on \bar{A}_1 ,

$$\liminf_{n \rightarrow +\infty} w_n \widehat{F}^n(v) \geq c > \Gamma_\delta \int v d\mu_\alpha.$$

This assumption together with (12) implies that, given $\xi > 0$, there exists $M = M(\xi) \in \mathbb{N}$ such that, for all $n \geq M$ and $x \in \bar{A}_1$,

$$\widehat{F}^n(v)(x) \geq e^{-\xi}\bar{\Gamma}_\delta^{-1}cn^{\delta-1}/l(n).$$

Thus, by Karamata's Tauberian Theorem for power series [6, Corollary 1.7.3], it follows that, for all $n \geq M$ and $x \in \bar{A}_1$,

$$\begin{aligned} \sum_{k=1}^n \widehat{F}^k(v)(x) &\geq \sum_{k=1}^M \widehat{F}^k(v)(x) + \sum_{k=M+1}^n e^{-\xi}\bar{\Gamma}_\delta^{-1}ck^{\delta-1}/l(k) \\ &\geq \sum_{k=1}^M \widehat{F}^k(v)(x) + e^{-\xi}\bar{\Gamma}_{1-\delta}\bar{\Gamma}_\delta^{-1}cn^\delta/l(n). \end{aligned}$$

Hence,

$$\liminf_{n \rightarrow +\infty} \frac{w_n}{n} \sum_{k=1}^n \widehat{F}^k(v)(x) = \liminf_{n \rightarrow +\infty} n^{-\delta}l(n)\bar{\Gamma}_\delta \sum_{k=1}^n \widehat{F}^k(v)(x) \geq \bar{\Gamma}_{1-\delta}\delta^{-1}c > \bar{\Gamma}_{1-\delta}\Gamma_\delta \int v d\mu_\alpha.$$

This is a contradiction, since by (12) and by combining Theorem 2.1 with Karamata's Tauberian Theorem for power series [6, Corollary 1.7.3], we have that the set \bar{A}_1 is a Darling-Kac set and therefore, by [1, Proposition 3.7.5], the α -Farey system is pointwise dual ergodic, meaning that, for μ_α -almost every $x \in [0, 1]$, we have that

$$\lim_{n \rightarrow +\infty} \frac{w_n}{n} \sum_{k=1}^n \widehat{F}^k(v)(x) = \bar{\Gamma}_{1-\delta}\Gamma_\delta \int v d\mu_\alpha.$$

□

Proof of Theorem 1.3 (iii). This follows from Propositions 4.9, 4.10, 4.11 and 4.13. □

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